

DSC 140A

Probabilistic Modeling & Machine Learning

Lecture 13 | Part 1

Parametric Density Estimation

Bayes Classifier

- ▶ Recall the **Bayes Classifier**: predict

$$\begin{cases} 1, & \text{if } \mathbb{P}(Y = 1 \mid \vec{X} = \vec{x}) > \mathbb{P}(Y = 0 \mid \vec{X} = \vec{x}), \\ 0, & \text{otherwise.} \end{cases}$$

- ▶ Equivalently, using **Bayes' rule**:

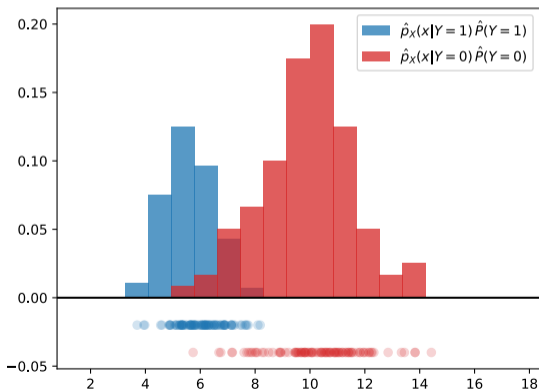
$$\begin{cases} 1, & \text{if } p_X(x \mid Y = 1)\mathbb{P}(Y = 1) > p_X(x \mid Y = 0)\mathbb{P}(Y = 0), \\ 0, & \text{otherwise.} \end{cases}$$

Estimating Densities

- ▶ We rarely know the true distribution.
- ▶ We must **estimate** it from data.
- ▶ When \vec{X} is continuous, we estimate **density**.

Last Time: Histogram Estimators

- **Histograms** provide one way of estimating densities.



Histogram Drawbacks

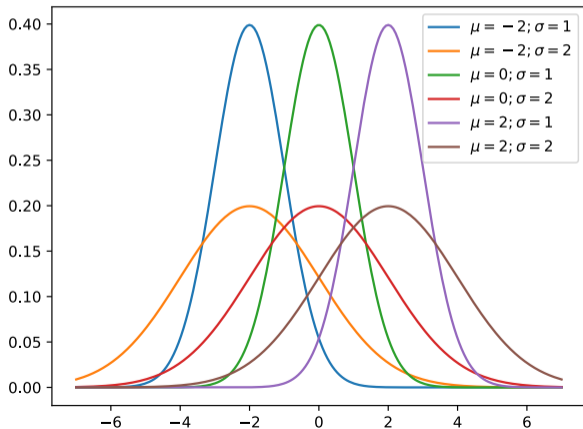
- ▶ We saw that histograms need massive amounts of data in high dimensions.
- ▶ The **Curse of Dimensionality**.

Observation

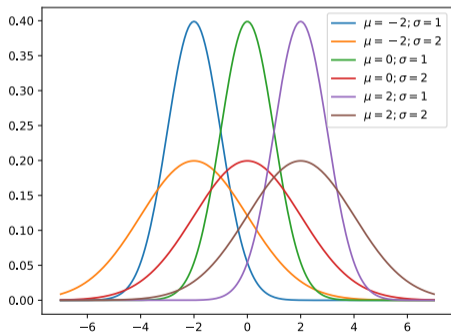
- ▶ Histogram estimators assume nothing about the **shape** of the true density.
- ▶ This makes them very flexible, but also data-hungry.
- ▶ **Idea:** Assume that the true, underlying density has a certain form.

Example: Gaussians

- ▶ Often assume that the true distribution is **Gaussian** (aka, **Normal**).



Example: Gaussians



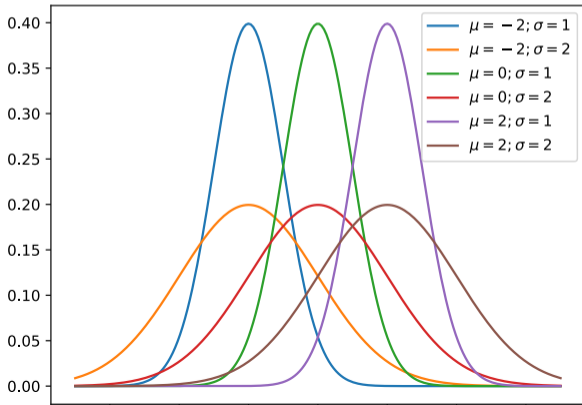
- ▶ **Recall:** the pdf of the Gaussian distribution:

$$p(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- ▶ μ and σ are **parameters**
 - ▶ μ controls center
 - ▶ σ controls width

Gaussian

- ▶ **Central Limit Theorem:** sums of independent random variables are Gaussian
- ▶ **Examples:** heights, measurement errors, test scores (kind of)



Parametric Distributions

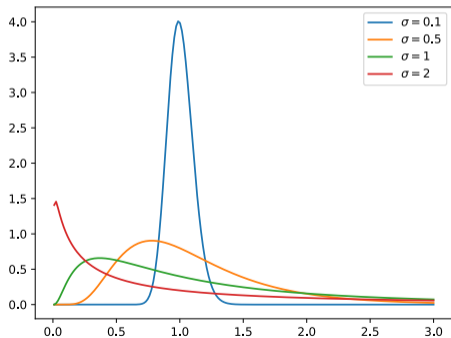
- ▶ A **parametric distribution** is **totally determined** by a finite number of **parameters**.
- ▶ **Example:** knowing μ and σ tells you everything about a Gaussian distribution.

Other Parametric Distributions

- ▶ There are many parametric distributions.
- ▶ **Discrete:** Bernoulli, Multinomial, Poisson, ...
- ▶ **Continuous:** Log-normal, Gamma, Pareto, ...

Example: Lognormal

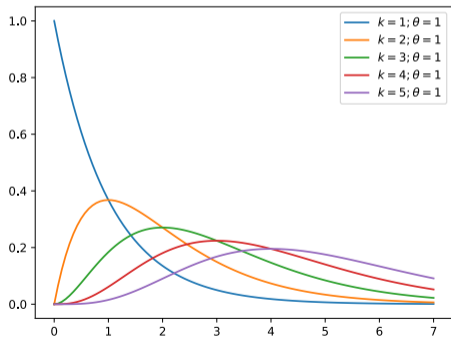
- ▶ Product of many independent positive random numbers.
- ▶ **Example:** length of comments in an internet forum



$$p(x; \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

Example: Gamma

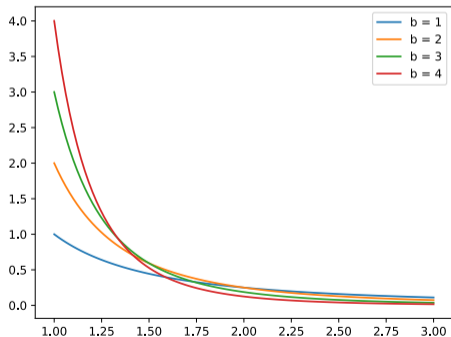
- ▶ **Examples:** wait times, size of rainfalls, insurance claims, ...



$$p(x; k, \theta) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-x/\theta}$$

Example: Pareto

- ▶ **Examples:** distribution of wealth, size of meteorites, ...



$$p(x; x_m, \alpha) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}}$$

Parametric Density Estimation

- ▶ In **parametric density estimation**, we assume data comes from some parametric density.
 - ▶ E.g., Gaussian, Log-Normal, Pareto, etc.
- ▶ But we don't know the parameters.
- ▶ Use data to **estimate** the parameters.

Non-Parametric Density Estimation

- ▶ Contrast this with estimating density with histograms.
- ▶ There were no parameters controlling the shape of the density.
- ▶ Histograms are **non-parametric** density estimators.

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Probabilistic Modeling & Machine Learning

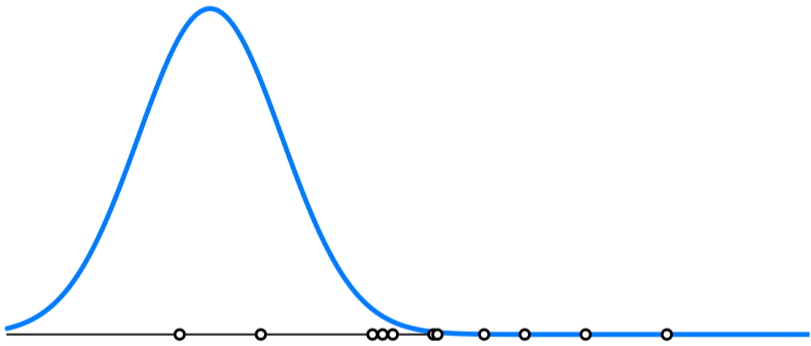
Lecture 13 | Part 2

Maximum Likelihood Estimation

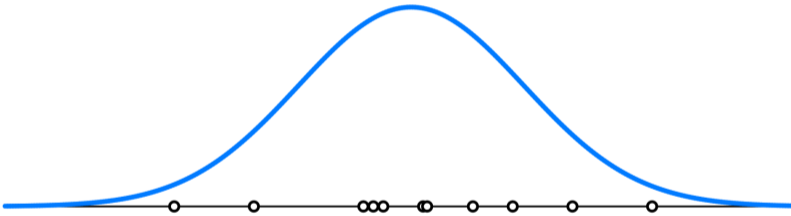
Parametric Density Estimation

- ▶ Suppose we have data $x^{(1)}, \dots, x^{(n)} \in \mathbb{R}$.
- ▶ Assume it came from a parametric distribution.
 - ▶ Say, a Gaussian.
- ▶ What were the parameter values used to generate the data?
- ▶ Using data to guess μ and σ is called **estimating** the parameters.





Unlikely



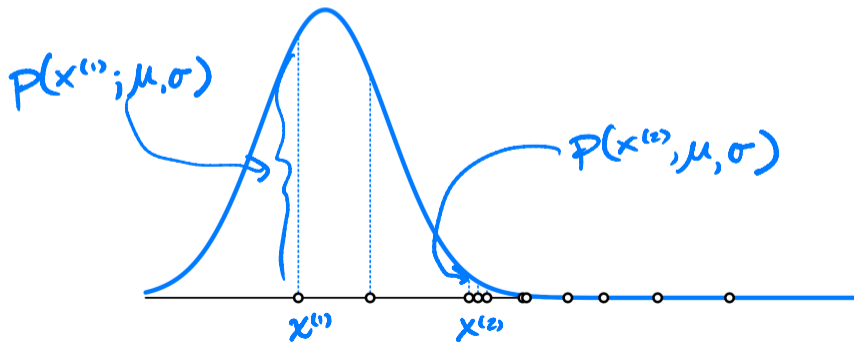
Likely

Intuition

- ▶ Some parameter choices seem **more likely** than others.
- ▶ That is, there is a greater chance that the data could have been generated by them.
- ▶ How can we quantify this?

Intuition

- ▶ Let p be the Gaussian probability density function.
- ▶ $p(x^{(i)}; \mu, \sigma)$ quantifies how likely it is to see $x^{(i)}$ if parameters μ and σ are used.



Exercise

Assume that $x^{(1)}, \dots, x^{(n)}$ are all sampled independently from a density with parameters μ, σ .

Think of $p(x^{(i)}; \mu, \sigma)$ as the “chance” of seeing $x^{(i)}$ under parameters μ and σ .

What is the chance of seeing $x^{(1)}$ and $x^{(2)}$ and $x^{(3)}$ and ... and $x^{(n)}$?

Intuition

- ▶ $p(x^{(1)}; \mu, \sigma) \times p(x^{(2)}; \mu, \sigma) \times \dots \times p(x^{(n)}; \mu, \sigma)$ quantifies likelihood of seeing $x^{(1)}, \dots, x^{(n)}$ simultaneously.
- ▶ In fact, it is the **joint density** of the data.
- ▶ But instead think of this as a function of μ and σ .

Likelihood

- ▶ The **likelihood** of μ and σ with respect to data $x^{(1)}, \dots, x^{(n)}$ is:

$$\begin{aligned}\mathcal{L}(\mu, \sigma; x^{(1)}, \dots, x^{(n)}) &= p(x^{(1)}; \mu, \sigma) \times p(x^{(2)}; \mu, \sigma) \times \dots \times p(x^{(n)}; \mu, \sigma) \\ &= \prod_{i=1}^n p(x^{(i)}; \mu, \sigma)\end{aligned}$$

Likelihood

- ▶ The likelihood function takes in parameters μ and σ and returns a real number.
- ▶ **Interpretation:** likelihood that data was generated by this choice of μ and σ .
- ▶ **Goal:** find μ and σ that **maximize** the likelihood.

<http://dsc140a.com/static/vis/mle/>

Maximizing Likelihood

- ▶ To maximize $\mathcal{L}(\mu, \sigma)$, we might take derivatives $\frac{\partial \mathcal{L}}{\partial \mu}$ and $\frac{\partial \mathcal{L}}{\partial \sigma}$, set to 0, solve.
- ▶ But the likelihood is often difficult to work with.

Example: Gaussian

- ▶ Assume that p is the Gaussian pdf.

$$p(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\frac{\partial \mathcal{L}}{\partial \mu} \quad \frac{\partial \mathcal{L}}{\partial \sigma}$$

- ▶ Then the likelihood function is:

$$\mathcal{L}(\mu, \sigma) = \prod_{i=1}^n \left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x^{(i)}-\mu)^2}{2\sigma^2}} \right)$$



Log Likelihood

- ▶ It is typically easier to work with the **log likelihood** instead.

$$\tilde{\mathcal{L}}(\mu, \sigma) = \ln \mathcal{L}(\mu, \sigma)$$

- ▶ **Fact:** Because $\ln x$ is **monotonically increasing**, a maximizer of $\ln \mathcal{L}$ also maximizes \mathcal{L}

Procedure: Gaussian

1. Write the log likelihood function $\tilde{\mathcal{L}}$.
2. Take derivatives $\partial\tilde{\mathcal{L}}/\partial\mu$ and $\partial\tilde{\mathcal{L}}/\partial\sigma$
3. Set to zero and solve for μ and σ .

Recall: Log Properties

- ▶ If a and b are positive: $\ln(a \times b) = \ln a + \ln b$
- ▶ If a and b are positive: $\ln(a/b) = \ln a - \ln b$
- ▶ If a is positive: $\ln a^p = p \ln a$

Step 1: Write Log Likelihood

- ▶ Write the log likelihood function for the Normal distribution.

$$P(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

$$\begin{aligned} \mathcal{L}(\mu, \sigma; x^{(1)}, \dots, x^{(n)}) &= \prod_{i=1}^n P(x^{(i)}; \mu, \sigma) \\ &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x^{(i)}-\mu)^2}{2\sigma^2}} \end{aligned}$$

Step 1: Write Log Likelihood

- Write the log likelihood function for the Normal distribution.

$$\mathcal{L}(\mu, \sigma; x^{(1)}, \dots, x^{(n)}) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x^{(i)} - \mu)^2}{2\sigma^2}}$$

$$\ln \mathcal{L}(\mu, \sigma) = \ln \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x^{(i)} - \mu)^2}{2\sigma^2}}$$

$$= \sum_{i=1}^n \ln \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x^{(i)} - \mu)^2}{2\sigma^2}}$$

$$= \sum_{i=1}^n \left(\ln \frac{1}{\sqrt{2\pi\sigma^2}} + \ln e^{-\frac{(x^{(i)} - \mu)^2}{2\sigma^2}} \right)$$

Step 1: Write Log Likelihood

- ▶ Write the log likelihood function for the Normal distribution.

$$\sum_{i=1}^n \left(\ln \frac{1}{\sqrt{2\pi\sigma^2}} + \cancel{\ln e}^{-\frac{(x^{(i)} - \mu)^2}{2\sigma^2}} \right) =$$

$$\sum_{i=1}^n \left[\cancel{\ln 1} - \ln \sigma \sqrt{2\pi} - \frac{(x^{(i)} - \mu)^2}{2\sigma^2} \right]$$

Step 2: Differentiate

- ▶ We have: $\tilde{\mathcal{L}} = \sum_{i=1}^n \left[-\ln \sigma - \ln \sqrt{2\pi} - \frac{(x^{(i)} - \mu)^2}{2\sigma^2} \right]$
- ▶ Compute $\partial \tilde{\mathcal{L}} / \partial \mu$:

$$\begin{aligned} \frac{\partial \tilde{\mathcal{L}}}{\partial \mu} &= \frac{\partial}{\partial \mu} \sum_{i=1}^n \left[-\ln \sigma - \ln \sqrt{2\pi} - \frac{(x^{(i)} - \mu)^2}{2\sigma^2} \right] \\ &= \sum_{i=1}^n \frac{\partial}{\partial \mu} \left[-\ln \sigma - \ln \sqrt{2\pi} - \frac{(x^{(i)} - \mu)^2}{2\sigma^2} \right] \\ &= \sum_{i=1}^n \frac{\partial}{\partial \mu} \left[-\frac{(x^{(i)} - \mu)^2}{2\sigma^2} \right] = + \sum_{i=1}^n \frac{2(x^{(i)} - \mu)}{2\sigma^2} = \sum_{i=1}^n \frac{(x^{(i)} - \mu)}{\sigma^2} \end{aligned}$$

Step 2: Differentiate

$$\frac{\partial}{\partial \sigma} \frac{1}{\sigma^2} = -\frac{2}{\sigma^3}$$

- ▶ We have: $\tilde{\mathcal{L}} = \sum_{i=1}^n \left[-\ln \sigma - \ln \sqrt{2\pi} - \frac{(x^{(i)} - \mu)^2}{2\sigma^2} \right]$
- ▶ Compute $\partial \tilde{\mathcal{L}} / \partial \sigma$:

$$\begin{aligned} \frac{\partial \tilde{\mathcal{L}}}{\partial \sigma} &= \frac{\partial}{\partial \sigma} \sum_{i=1}^n \left[-\ln \sigma - \ln \sqrt{2\pi} - \frac{(x^{(i)} - \mu)^2}{2\sigma^2} \right] \\ &= \sum_{i=1}^n \frac{\partial}{\partial \sigma} \left[-\ln \sigma - \ln \sqrt{2\pi} - \frac{(x^{(i)} - \mu)^2}{2\sigma^2} \right] \\ &= \sum_{i=1}^n \left[-\frac{1}{\sigma} + \frac{(x^{(i)} - \mu)^2}{2} \frac{(-2)}{\sigma^3} \right] \end{aligned}$$

$$\sum (a_i + b_i) = \sum a_i + \sum b_i$$

Step 3: Solve

- ▶ We have $\partial \tilde{L} / \partial \mu = \frac{1}{\sigma^2} \sum_{i=1}^n (x^{(i)} - \mu)$
- ▶ Solve $\partial \tilde{L} / \partial \mu = 0$ for μ .

$$\frac{1}{\sigma^2} \sum_{i=1}^n (x^{(i)} - \mu) = 0 \Rightarrow \sum_{i=1}^n (x^{(i)} - \mu) = 0$$

$$\Rightarrow \sum_{i=1}^n x^{(i)} - \sum_{i=1}^n \mu = 0$$

$$\Rightarrow \sum_{i=1}^n x^{(i)} = \sum_{i=1}^n \mu$$

$$\Rightarrow \sum_{i=1}^n x^{(i)} = n\mu \Rightarrow \mu_{ME} = \frac{1}{n} \sum_{i=1}^n x^{(i)}$$

Step 3: Solve

- ▶ We have $\partial \tilde{L} / \partial \sigma = \sum_{i=1}^n \left[-\frac{1}{\sigma} + \frac{(x^{(i)} - \mu)^2}{\sigma^3} \right]$
- ▶ Solve $\partial \tilde{L} / \partial \sigma = 0$ for σ .

$$\sum_{i=1}^n \left(-\frac{1}{\sigma} + \frac{(x^{(i)} - \mu_{MLE})^2}{\sigma^3} \right) = 0$$

$$\Rightarrow -\sum \frac{1}{\sigma} + \frac{1}{\sigma^3} \sum (x^{(i)} - \mu_{MLE})^2 = 0$$

$$\Rightarrow \frac{1}{\sigma^2} \sum (x^{(i)} - \mu_{MLE})^2 = n$$

$$\sqrt{\frac{1}{n} \sum (x^{(i)} - \mu_{MLE})^2} = \sigma_{MLE}$$

MLEs for Gaussian Distribution

- ▶ We have found the **maximum likelihood estimates** for the Gaussian distribution:

$$\mu_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^n x^{(i)} \quad \sigma_{\text{MLE}} = \sqrt{\frac{1}{n} \sum_{i=1}^n (x^{(i)} - \mu_{\text{MLE}})^2}$$

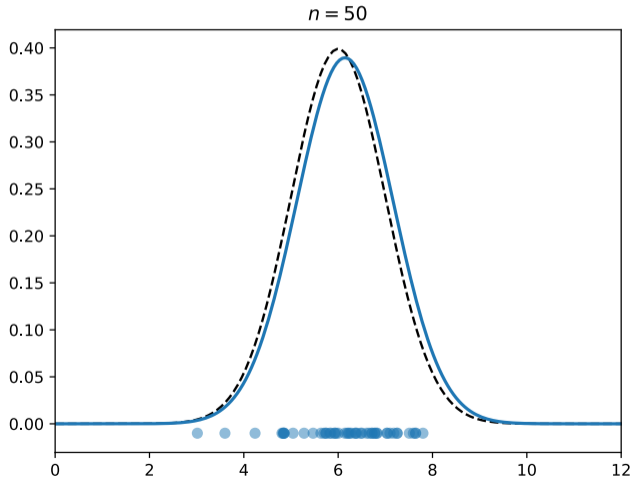
“Fitting” a Gaussian

- ▶ Suppose we wish to “fit” a Gaussian to data $x^{(1)}, \dots, x^{(n)}$.
- ▶ The **maximum likelihood** approach:
 1. Compute:

$$\mu_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^n x^{(i)} \quad \sigma_{\text{MLE}} = \sqrt{\frac{1}{n} \sum_{i=1}^n (x^{(i)} - \mu_{\text{MLE}})^2}$$

2. Use these as parameters of the Gaussian.

Example



In General

- ▶ **Maximum Likelihood Estimation** (MLE) can be used for a variety of densities.
- ▶ Suppose density p has parameters $\theta_1, \dots, \theta_k$

1. Write log likelihood function:

$$\ln \mathcal{L}(\theta_1, \dots, \theta_k) = \sum_{i=1}^n \ln p(x^{(1)}, \dots, x^{(n)}; \theta_1, \dots, \theta_k)$$

2. Compute derivatives: $\partial \tilde{\mathcal{L}} / \partial \theta_1, \partial \tilde{\mathcal{L}} / \partial \theta_2, \dots, \partial \tilde{\mathcal{L}} / \partial \theta_k$
3. Set derivatives to zero, solve for $\theta_1, \dots, \theta_k$.

In Practice

- ▶ The MLE for a parameter only needs to be derived once.
- ▶ Many textbooks, statistics packages, and Wikipedia list the MLE parameter estimators.

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Probabilistic Modeling & Machine Learning

Lecture 13 | Part 3

Parametric vs. Non-Parametric Density Estimation

Making Predictions

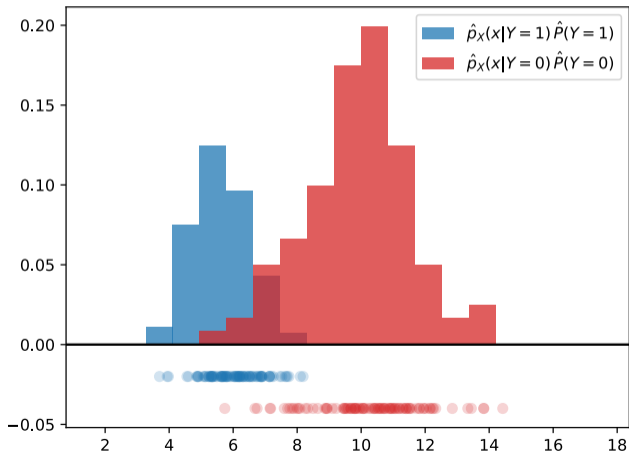
- ▶ We observe a data set $\{(x^{(i)}, y_i)\}$ of flipper lengths and penguin species (0 or 1).
- ▶ **Task:** Given the flipper length of a new penguin, what is its species?
- ▶ Bayes' classifier: predict
$$\begin{cases} 1, & \text{if } p_X(x | Y = 1)\mathbb{P}(Y = 1) > p_X(x | Y = 0)\mathbb{P}(Y = 0), \\ 0, & \text{otherwise.} \end{cases}$$

Estimating Densities

- ▶ We must estimate $p_X(x | Y = 0)$ and $p_X(x | Y = 1)$.
- ▶ Approach 1: Non-parametric (histograms)
- ▶ Approach 2: Parametric

Approach 1: Non-Parametric

- ▶ Estimate $p_X(x | Y = 0)$ and $p_X(x | Y = 1)$ with histograms.

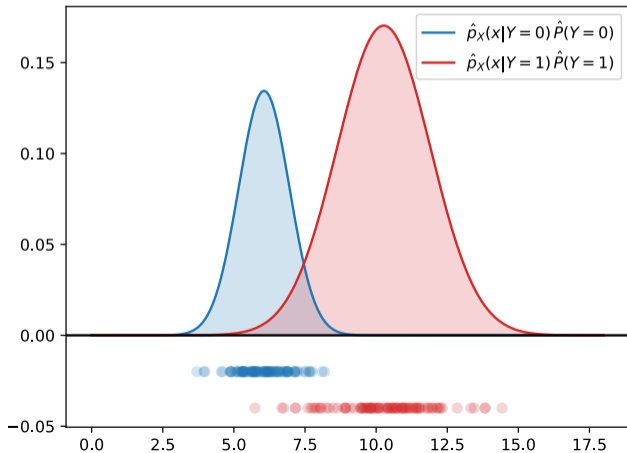


Approach 2: Parametric

- ▶ Must choose a parametric distribution.
- ▶ Plotting a histogram, data looks roughly normal.
- ▶ We will fit Gaussians.

Approach 2: Parametric

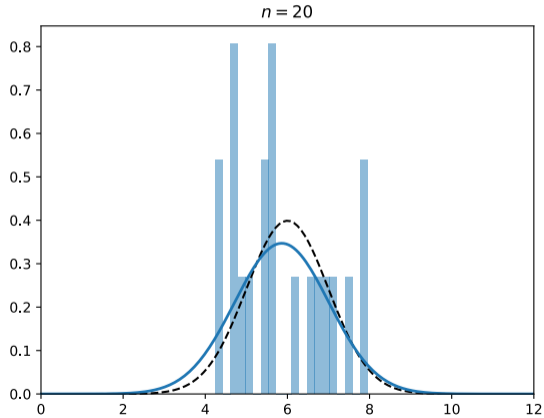
- ▶ Estimate $p_X(x | Y = 0)$ and $p_X(x | Y = 1)$ by fitting Gaussians with MLE.



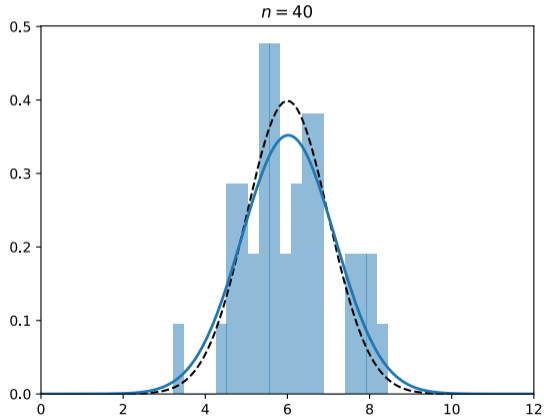
Data Requirements

- ▶ Suppose the underlying distribution that produced the data actually was a Gaussian.
 - ▶ Or close to one.
- ▶ The parametric approach will require less data than the non-parametric.

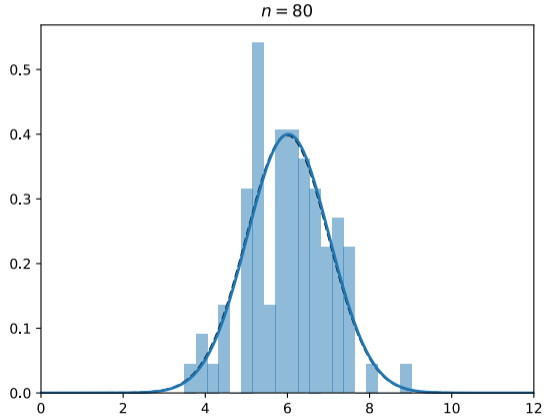
Data Requirements



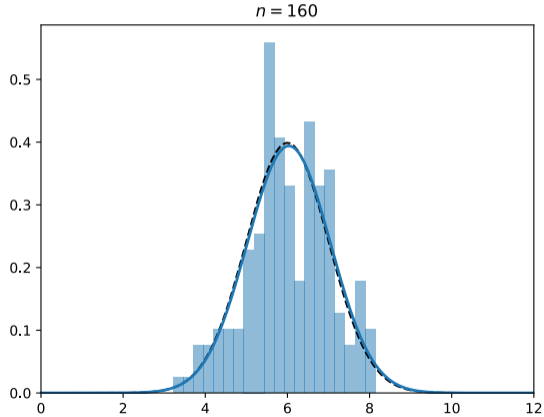
Data Requirements



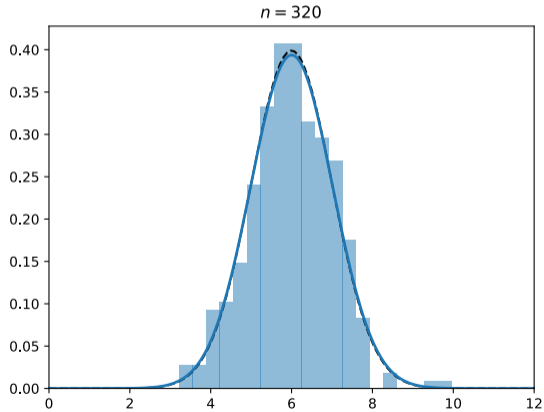
Data Requirements



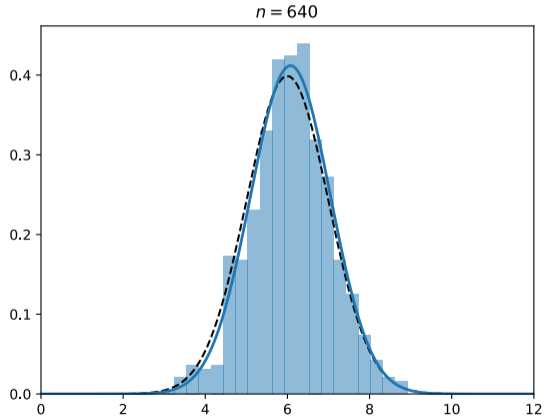
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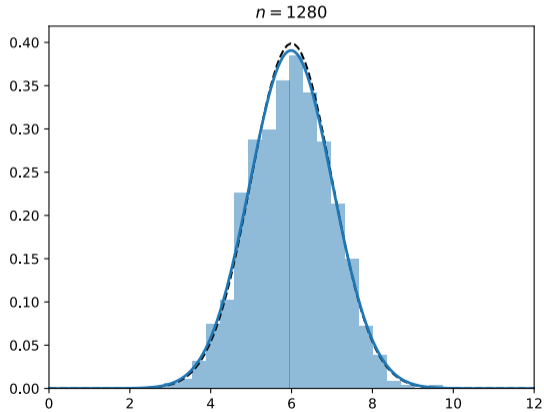
Data Requirements



Data Requirements



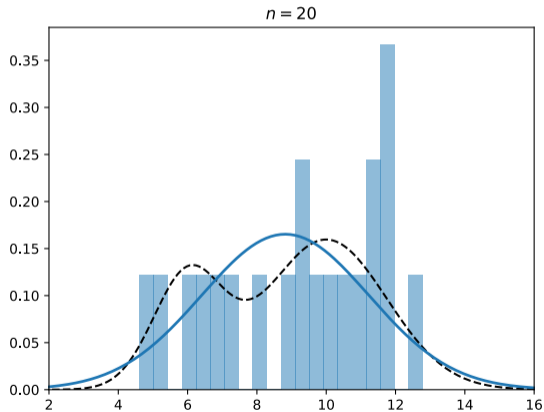
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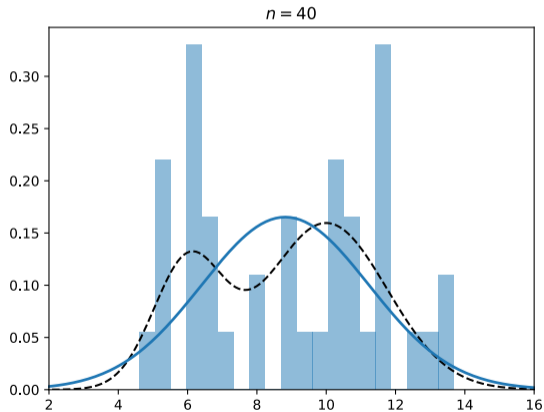
Mis-specification

- ▶ However, suppose the underlying distribution is **not** Gaussian.
- ▶ No amount of data will allow the parametric approach to get close.
 - ▶ The model has been **mis-specified**.
- ▶ But the non-parametric approach will be close, eventually.

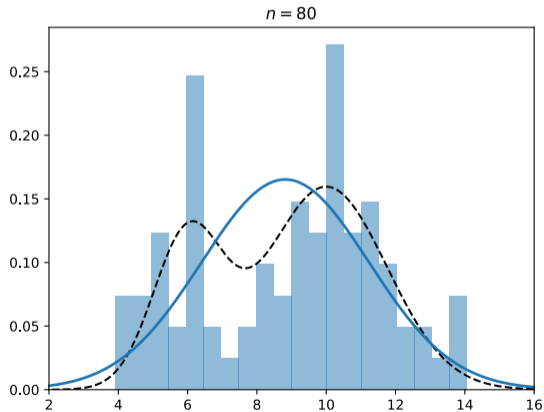
Parametric vs. Non-Parametric



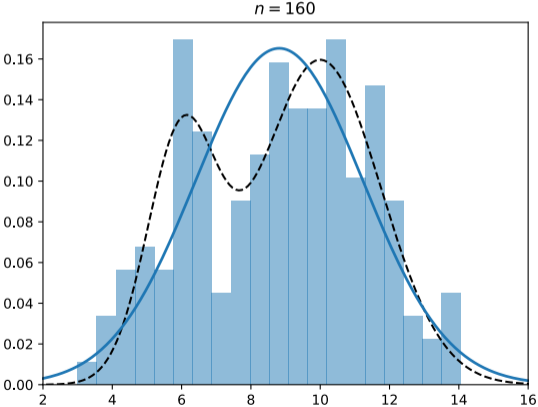
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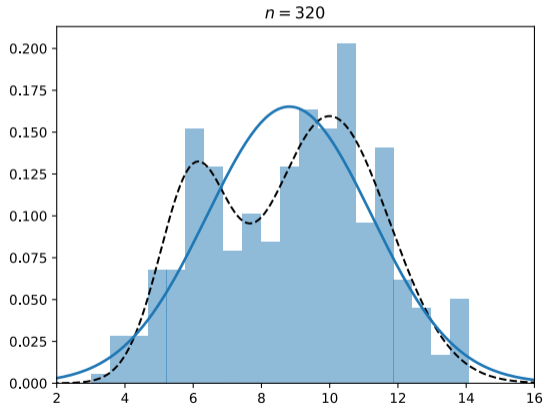
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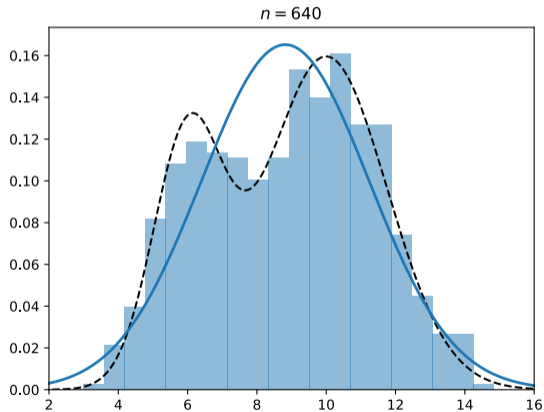
Parametric vs. Non-Parametric



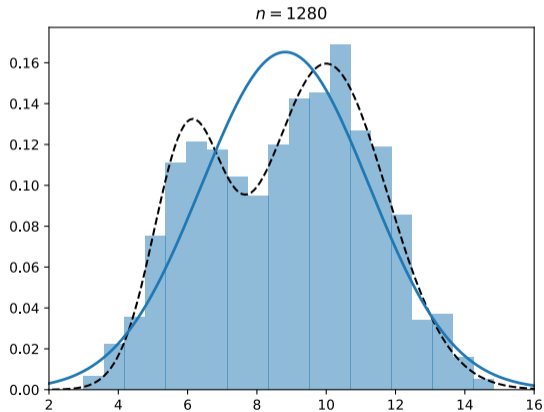
Parametric vs. Non-Parametric



Parametric vs. Non-Parametric



Parametric vs. Non-Parametric



High Dimensions

- ▶ Non-parametric approaches can fit arbitrary densities, but they require lots of data.
 - ▶ Especially in high dimensions!
- ▶ Parametric approaches require less data, provided that they are correctly specified.
- ▶ **Next time:** parametric density estimation in high dimensions.