DSC 140A - Discussion 06

Problem 1.

In lecture, we defined a kernel function to be a function k which computes the dot product of vectors after they are mapped to some high-dimensional space. The useful thing about kernel functions is that they allow us to compute these dot products without actually mapping vectors them to the high-dimensional space, which can be costly. In this problem, we will consider the the 2nd-order *polynomial kernel*, defined to be

$$k(\vec{x}, \vec{x}') = (1 + \vec{x} \cdot \vec{x}')^2.$$

Let $\vec{\phi}(\vec{x}) : \mathbb{R}^3 \to \mathbb{R}^{10}$ be the mapping:

$$\vec{\phi}(\vec{x}) = (1, x_1^2, x_2^2, x_3^2, \sqrt{2}\, x_1, \sqrt{2}\, x_2, \sqrt{2}\, x_3, \sqrt{2}\, x_1 x_2, \sqrt{2}\, x_1 x_3, \sqrt{2}\, x_2 x_3)^T,$$

where x_1, x_2, x_3 are the components of the input vector, $\vec{x}$. That is, $\vec{\phi}$ is a feature map which maps a vector into a higher-dimensional space.

Show that $k(\vec{x}, \vec{y}) = \vec{\phi}(\vec{x}) \cdot \vec{\phi}(\vec{y})$. That is, that k indeed computes the inner product of vectors in the higher-dimensional space.

Solution:

$$
\begin{aligned}
k(\vec{x}, \vec{y}) &= (1 + x_1 y_1 + x_2 y_2 + x_3 y_3)^2 \\
&= 1 + x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2 + 2x_1 y_1 + 2x_2 y_2 + 2x_3 y_3 + 2x_1 x_2 y_2 y_1 + 2x_1 x_3 y_3 y_1 + 2x_2 x_3 y_2 y_3
\end{aligned}
$$

Now,

$$
\vec{\phi}(\vec{x}) \cdot \vec{\phi}(\vec{y}) = (1, x_1^2, x_2^2, x_3^2, \sqrt{2}\, x_1, \sqrt{2}\, x_2, \sqrt{2}\, x_3, \sqrt{2}\, x_1 x_2, \sqrt{2}\, x_1 x_3, \sqrt{2}\, x_2 x_3)
\begin{pmatrix}
1 \\
y_1^2 \\
y_2^2 \\
y_3^2 \\
\sqrt{2}\, y_1 \\
\sqrt{2}\, y_2 \\
\sqrt{2}\, y_3 \\
\sqrt{2}\, y_1 y_2 \\
\sqrt{2}\, y_1 y_3 \\
\sqrt{2}\, y_2 y_3
\end{pmatrix}
$$

$$
= 1 + x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2 + 2x_1 y_1 + 2x_2 y_2 + 2x_3 y_3 + 2x_1 x_2 y_2 y_1 + 2x_1 x_3 y_3 y_1 + 2x_2 x_3 y_2 y_3
$$

Therefore, they are equivalent.