DSC 140A - Homework 03

Due: Wednesday, April 24

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Unless otherwise noted by the problem's instructions, show your work or provide some justification for your answer. Homeworks are due via Gradescope at 11:59 PM.

Problem 1.

Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is convex and $g : \mathbb{R} \to \mathbb{R}$ is convex and non-decreasing. That is, if $a > b$, then $g(a) \geq g(b)$.

Show that the composition of these functions, $h(\vec{x}) = g(f(\vec{x}))$, is also convex.

Hint: you'll want to go back to the definition to show this is true. A similar problem was solved in discussion.

Problem 2.

The *hinge loss* is defined to be

$$L_{\text{hinge}}(\vec{w}, \vec{x}, y) = \max\{0, 1 - y\, \vec{w} \cdot \text{Aug}(\vec{x})\}$$

Consider the below quantity, which is called the *regularized* empirical risk:

$$R(\vec{w}) = \frac{C}{n} \sum_{i=1}^{n} L_{\text{hinge}}(\vec{w}, \vec{x}^{(i)}, y_i) + \|\vec{w}\|^2$$

Here, C is a positive constant, n is the number of data points, $\vec{x}^{(i)}$ is the ith data point, y_i is the label of the ith data point, and $\vec{w}$ is a vector of weights.

Show that $R(\vec{w})$ is a convex function of $\vec{w}$.

(We will discuss regularization and the hinge loss in future lectures; for this problem it's not necessary to know anything about them apart from the definitions given above.)

Hint: you will probably *not* want to use the formal definition of convexity here. Instead, you'll want to show that R is composed of simpler functions which themselves are convex.

Problem 3.

Consider a linear prediction function H used for binary classification, and assume that when the output of H is positive we predict for class +1, and when it's negative we predict for class -1. This means that the **decision boundary** is where $H(\vec{x}) = 0$.

In lecture, we saw that a linear prediction function has the form:

$$\begin{aligned} H(\vec{x}) &= w_0 + w_1 x_1 + \ldots + w_d x_d \\ &= \vec{w} \cdot \text{Aug}(\vec{x}) \end{aligned}$$

where $\vec{w} = (w_0, w_1, \ldots, w_d)^T$. In this problem, it will also be useful to define the vector $\vec{w}' = (w_1, \ldots, w_d)^T$, which is the same as $\vec{w}$ except that it does not include w_0. Note that $\vec{x} \cdot \vec{w}' = w_1 x_1 + \ldots w_d x_d$. With this definition, we can write $H(\vec{x})$ in a slightly different way:

$$H(\vec{x}) = w_0 + \vec{w}' \cdot \vec{x}$$

Remember this formula, as it will be useful several times below!

Over the course of this problem, we'll answer the question: how is the magnitude of $H(\vec{x})$ related to the distance between $\vec{x}$ and the decision boundary?

Note: for this problem it may be useful to review the properties of the dot product and vector algebra. In particular, remember that $\|u\|$ denotes the norm (length) of a vector, and that $\vec{u} \cdot \vec{u} = \|u\|^2$. By dividing a vector by its norm, as in $\vec{u}/\|\vec{u}\|$, we obtain a *unit vector* with unit length – a unit vector is useful for specifying a direction. To find the component of a vector $\vec{u}$ that points in the same direction as another vector $\vec{v}$, we write $\vec{u} \cdot \vec{v}/\|\vec{v}\|$. Also remember that $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$, and that $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

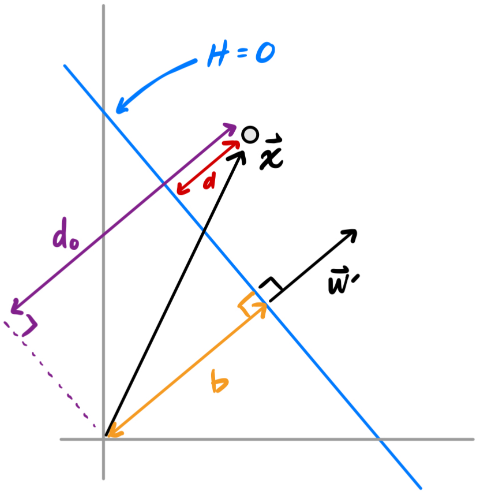
a) Show that for any point $\vec{z}$ on the decision boundary, $\vec{w}' \cdot \vec{z} = -w_0$.

Hint: use that fact that $H(\vec{z}) = 0$.

b) Argue that $\vec{w}'$ is orthogonal to the decision boundary.

Hint: take two arbitrary points $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ that are assumed to be on the decision boundary. Then, since we know that the boundary is linear (it is a line, plane, etc.), the difference of these vectors, $\vec{\delta} = \vec{x}^{(1)} - \vec{x}^{(2)}$ is parallel to the boundary. To show that $\vec{w}'$ is orthogonal to the boundary, it suffices to show that $\vec{w}' \cdot \vec{\delta} = 0$. Make sure you somehow use the fact that $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are on the decision boundary.

c) Now that we know that $\vec{w}'$ is orthogonal to the decision boundary, we can draw a better picture of the situation:



The blue line is the decision boundary – it is where $H = 0$. We have drawn an arbitary point $\vec{x}$, along with several distances:

- d: the (signed) distance from the decision boundary to $\vec{x}$
- b: the distance from the the origin to the decision boundary
- d_0: the length of the component of $\vec{x}$ that is orthogonal to the decision boundary.

We're most interested in knowing d. First, though, we need to find b.

Consider the vector $b\vec{w}'/\|\vec{w}'\|$; this is a vector from the origin to the decision boundary that is orthogonal to the boundary and with length b.

Since this vector is on the decision boundary, $H(b\vec{w}'/\|\vec{w}'\|) = 0$. Using this fact, show that $b = -\dfrac{w_0}{\|\vec{w}'\|}$.

Hint: first show that $H(b\vec{w}'/\|\vec{w}'\|) = b\|\vec{w}'\| + w_0$, then set this to zero and solve for b.

d) Recall that d_0 is the component of $\vec{x}$ that is orthogonal to the decision boundary; this is simply $\vec{x} \cdot \vec{w}'/\|\vec{w}'\|$. From the picture, $d_0 = d + b$. We know d_0 and b, and can therefore solve for d.

Use this to show that $|d| = |H(\vec{x})|/\|\vec{w}'\|$.

We have shown that the distance between $\vec{x}$ and the decision boundary is proportional to the output of the prediction function, $H(\vec{x})$. This gives us a very useful interpretation of $|H(\vec{x})|$! For example, this means if $H(\vec{x}^{(1)}) > H(\vec{x}^{(2)}) > 0$, then $\vec{x}^{(1)}$ is further from the decision boundary than $\vec{x}^{(2)}$.

Problem 4.

The file below contains data suitable for a binary classification problem. `https://f000.backblazeb2.com/file/jeldridge-data/003-two_clusters/data.csv`

The file contains three columns: x_1, x_2, and y. The first two columns are the features, and the third column is the label.

a) Write a function to compute the empirical risk for the perceptron loss, $R_{\text{tron}}(\vec{w})$, with respect to the data and the parameters $\vec{w}$.

Use your function to compute the empirical risk for the vector $(1, 1, 1)^T$, and report the result. Include your code.

b) Write a function to compute the subgradient of the empirical risk for the perceptron loss, and use your code to compute the subgradient at the point $(1, 1, 1)^T$. Report the result.

c) Run subgradient descent to train a perceptron on the data. Use the initial vector $(1, 1, 1)^T$. Report the learned parameter vector.

d) Plot the data and the decision boundary of the perceptron you learned above. Each point should be colored according to its label.

Hint: There are two ways to plot the decision boundary. One way is to use the fact that $\text{Aug}(\vec{x}) \cdot \vec{w} = 0$ for points on the decision boundary to solve for x_2 in terms of x_1, giving you the equation of a line. Another approach is to use `plt.contour` to plot the contours of the prediction function, $H(\vec{x})$; in particular, we plot the contour where $H(\vec{x}) = 0$. The second approach is more general and can be used for any prediction function, but you'll want to read the docs to understand how to use it.